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# Analytical solutions describing the consolidation of a multi-layered soil under circular loading

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**Abstract** Analytical solutions describing the consolidation of a multi-layered soil under circular loading are presented. From the governing equations of saturated poroelastic soil in a cylindrical coordinate system, the eighth-order state-space equation of consolidation is obtained by eliminating the variation of time $t$ using the Laplace transform together with the technique of Fourier expansions with respect to the coordinate $\theta$ and the Hankel transform with respect to coordinate $r$. The solution of the eighth-order state-space equation is derived directly by using the Laplace transform and its inversion of the $z$-domain. Based on the continuity between layers and the boundary conditions, the transfer-matrix method is utilized to derive the solutions for the consolidation of a multi-layered soil under circular loading in the transformed domain. By the inversion of the Laplace transform and the Hankel transform, the analytical solutions in the physical domain are obtained. A numerical analysis based on the solutions is carried out by a corresponding program.

**Keywords** Biot consolidation · Circular loading · Fourier expansions · Integral transform · Transfer matrix

## 1 Introduction

Consolidation of a saturated soil is an essential problem in geotechnical engineering. Due to a complex interactive process between a solid phase (soil skeleton) and a liquid phase (pore water) in a soil, many efforts have been made by researchers to evaluate the consolidation of a saturated soil. Biot [1] developed a theory which can reasonably describe the complicated relationship between the stress in an elastic body and the water flow in pores (i.e., accurately reflect time-dependent behavior of soils). Therefore, this theory has been adopted by many researchers to develop the solutions to describe consolidation of saturated soils. However, owing to the complexity of the poroelastic governing equations, it is usually a challenge to find an analytical solution based on Biot's consolidation theory, except for cases with simple geometries.

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Despite this difficulty, however, some systematic analytical solutions have been established in the past to deal with the consolidation of saturated soils. Most of previous research work has been focused on a uniform half-space medium, such as that of McNamee and Gibson [2,3], Schiffman and Funguroli [4], and Yue and Selvadurai [5]. However, there exist a few studies on a finite soil layer and a multi-layered soil. For example, Gibson et al. [6] and Booker [7] proposed analytical solutions for the consolidation of a finite soil layer subjected to surface loading. Vardoulakis and Harnpattananich [8] adopted a numerical method with displacement functions and integral transforms to analyze three-dimensional consolidation problems. Booker and Small [9–11] and Mei et al. [12] used the finite-layer method to analyze the consolidation of a multi-layered soil. Senjuntichai and Rajapakse [13] solved the quasi-statics of a multi-layered poroelastic medium using the exact stiffness method. Pan [14] employed the propagator-matrix method with vector functions to obtain a Green's function in a layered poroelastic half-space. Wang and Fang [15,16] used the state-vector method to analyze axisymmetric and non-axisymmetric Biot consolidation problems for multi-layered porous media. Ai and Han [17] used the transfer-matrix method to obtain the solution to the plane-strain consolidation of a multi-layered soil. In addition, numerical methods, such as the finite-element method [18] and the boundary-element method [19], have been used to solve more complicated consolidation problems.

Among all the above-mentioned methods, the transfer-matrix method is one of the most efficient and accurate methods available for solving elastostatic and poroelastic problems in multi-layered materials [17,20–22] and laminated composite plates and shells [23–30], and has an obvious advantage that the size of the final equation system is independent of the number of layers in the system. For example, a plane-strain consolidation problem of an $n$-layered poroelastic medium can be solved using only $3 \times 3$ linear equations [17]. In this paper, Biot's consolidation problem for a multi-layered soil under circular loading in a cylindrical coordinate system is investigated by the transfer-matrix method. The transfer-matrix method for plane-strain and three-dimensional Biot consolidation problems for a multi-layered soil in a Cartesian coordinate system was developed and presented in [17,22]. Obviously, these solutions are suitable for strip, square, or rectangular loading. However, the shape of the loading can be circular in practice, such as a tank foundation; therefore, it is needed for the development of a solution to the consolidation problem regarding a multi-layered soil under circular loading in a cylindrical coordinate system. Even though the general approach is similar to that in a Cartesian coordinate system, the derivations of the new solution in a cylindrical coordinate system are quite different. In the development of the solutions in a Cartesian coordinate system [22], the Laplace transform with respect to time $t$ and the double Fourier transform with respect to coordinates $x$ and $y$ were utilized to obtain the ordinary-differential matrix equation in the transformed domain. In this study, the technique of the Laplace transform with respect to time $t$ is still used; however, the variables for the displacements, stresses, excess pore pressure, and flux are expressed using the Fourier expansions suggested by Muki [31] for elastostatics to eliminate the coordinates $\theta$ and $r$ by means of the Hankel transform. In addition, the solution in the transformed domain of the ordinary-differential matrix equation in the Cartesian coordinate system [22] was developed based on the theorem of Cayley–Hamilton [32], which involved a complex work load. In this study the solution in the transformed domain of the ordinary-differential matrix equation is derived directly using the Laplace transform and its inversion [33] of the $z$-domain.

The main objectives of this study can be summarized as follows: (1) to derive the $8 \times 8$ transfer matrix for Biot's consolidation in the cylindrical coordinate system and use the transfer-matrix method to obtain the solution to Biot's consolidation problem of a multi-layered soil under circular loading and (2) to provide an alternative way to develop analytical solutions to Biot's consolidation problem in a cylindrical coordinate system and propose fundamental solutions to analyze the interaction between saturated multi-layered soils and foundation structures.

## 2 Transfer matrix for a single soil layer

When the body force in a single soil layer is ignored, the equations of equilibrium are:

$$\frac{\partial \sigma_{rz}}{\partial z} + \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{1a}$$

$$\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta}+\frac{2\sigma_{r\theta}}{r}=0, \tag{1b}$$

$$\frac{\partial \sigma_{z}}{\partial z}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}=0, \tag{1c}$$

where $\sigma_r$, $\sigma_\theta$, and $\sigma_z$ are the normal stresses acting on the plane normal to the $r$, $\theta$, and $z$ axes, respectively.

According to the principle of effective stresses and Hooke's law, we have:

$$\sigma_r+\sigma=M\frac{\partial u_r}{\partial r}+\lambda\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+\lambda\frac{\partial u_z}{\partial z},\quad \sigma_\theta+\sigma=\lambda\frac{\partial u_r}{\partial r}+M\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+\lambda\frac{\partial u_z}{\partial z}, \tag{2a,b}$$

$$\sigma_z+\sigma=\lambda\frac{\partial u_r}{\partial r}+\lambda\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+M\frac{\partial u_z}{\partial z},\quad \sigma_{rz}=G\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right), \tag{2c,d}$$

$$\sigma_{\theta z}=G\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right),\quad \sigma_{r\theta}=G\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right), \tag{2e,f}$$

where $\sigma$ is the excess pore pressure (positive under compression); $u_r$, $u_\theta$ and $u_z$ are the displacements in the $r$, $\theta$, and $z$ directions, respectively; $\lambda=\frac{vE}{(1+v)(1-2v)}$, $M=\lambda+2G=\frac{E(1-v)}{(1+v)(1-2v)}$, and $E$, $G$, $v$ are Young's modulus, the shear modulus, and Poisson's ratio of the soil, respectively.

The continuity equation for a poroelastic medium is:

$$\frac{\partial e}{\partial t}=\frac{k}{\gamma_w}\cdot\nabla^2\sigma, \tag{3}$$

where $e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}$ is the volumetric strain, $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian operator, $k$ is the coefficient of the permeability, and $\gamma_w$ is the unit weight of water. According to Darcy's law, the flux in the $z$-direction can be defined as, $Q=\frac{k}{\gamma_w}\frac{\partial \sigma}{\partial z}$.

The displacements, the stresses, the excess pore pressure, and the flux can be expressed in Fourier expansions as suggested by Muki [31] for elastostatics, i.e.,

$$\begin{aligned}
u_r&=\sum_{m=0}^{\infty}u_{rm}\cos m\theta,\quad \sigma_{rz}=\sum_{m=0}^{\infty}\sigma_{rzm}\cos m\theta,\\
u_\theta&=\sum_{m=0}^{\infty}u_{\theta m}\sin m\theta,\quad \sigma_{\theta z}=\sum_{m=0}^{\infty}\sigma_{\theta zm}\sin m\theta,\\
u_z&=\sum_{m=0}^{\infty}u_{zm}\cos m\theta,\quad \sigma_{z}=\sum_{m=0}^{\infty}\sigma_{zm}\cos m\theta,\\
\sigma&=\sum_{m=0}^{\infty}\sigma_{m}\cos m\theta,\quad Q=\sum_{m=0}^{\infty}Q_{m}\cos m\theta.
\end{aligned} \tag{4}$$

When $m=0$, all the variables mentioned in Eq. 4 are independent of coordinate $\theta$, and the problem in this study is degenerated into an axis-symmetric one.

The Laplace transform and its inversion [33] are defined as:

$$\tilde{f}(r,\theta,z,s)=\int_0^\infty f(r,\theta,z,t)\mathrm{e}^{-st}\mathrm{d}t,\quad f(r,\theta,z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\tilde{f}(r,\theta,z,s)e^{st}\mathrm{d}s,$$

where $s$ denotes the Laplace-transform parameter of $t$.

Applying the Laplace transform to Eqs. 1–4 and assuming zero initial volumetric strain $e$ everywhere, we obtain the following equations after some algebraic manipulations:

$$\frac{\partial \tilde{u}_{rm}}{\partial z}=-\frac{\partial \tilde{u}_{zm}}{\partial r}+\frac{1}{G}\tilde{\sigma}_{rzm},\quad \frac{\partial \tilde{u}_{\theta m}}{\partial z}=m\frac{1}{r}\tilde{u}_{zm}+\frac{1}{G}\tilde{\sigma}_{\theta zm}, \tag{5a,b}$$

$$\frac{\partial \tilde{u}_{zm}}{\partial z} = \frac{-v}{1-v}\left(\frac{\partial \tilde{u}_{rm}}{\partial r} + \frac{\tilde{u}_{rm}}{r}\right) - \frac{v}{1-v} m \frac{1}{r}\tilde{u}_{\theta m} + \frac{1-2v}{2G(1-v)}(\tilde{\sigma}_{zm} + \tilde{\sigma}_m),$$
$$\frac{\partial \tilde{\sigma}_{zm}}{\partial z} = -m\frac{1}{r}\tilde{\sigma}_{\theta zm} - \frac{\partial \tilde{\sigma}_{rzm}}{\partial r} - \frac{\tilde{\sigma}_{rzm}}{r}, \tag{5c,d}$$

$$\begin{aligned}\frac{\partial \tilde{\sigma}_{rzm}}{\partial z} =& -\frac{4G(\lambda+G)}{M}\frac{\partial^2 \tilde{u}_{rm}}{\partial r^2} - \frac{4G(\lambda+G)}{M}\frac{1}{r}\frac{\partial \tilde{u}_{rm}}{\partial r} + \frac{4G(\lambda+G)}{M}\frac{1}{r^2}\tilde{u}_{rm} + m^2 G\frac{1}{r^2}\tilde{u}_{rm} \\ & - m\frac{G(3\lambda+2G)}{M}\frac{1}{r}\frac{\partial \tilde{u}_{\theta m}}{\partial r} + m\frac{G(5\lambda+6G)}{M}\frac{1}{r^2}\tilde{u}_{\theta m} + \frac{2G}{M}\frac{\partial \tilde{\sigma}_m}{\partial r} - \frac{\lambda}{M}\frac{\partial \tilde{\sigma}_{zm}}{\partial r},\end{aligned} \tag{5e}$$

$$\begin{aligned}\frac{\partial \tilde{\sigma}_{\theta zm}}{\partial z} =& -G\frac{\partial^2 \tilde{u}_{\theta m}}{\partial r^2} - G\frac{1}{r}\frac{\partial \tilde{u}_{\theta m}}{\partial r} + m\frac{G(3\lambda+2G)}{M}\frac{1}{r}\frac{\partial \tilde{u}_{rm}}{\partial r} + m\frac{G(5\lambda+6G)}{M}\frac{1}{r^2}\tilde{u}_{rm} \\ & + G\frac{1}{r^2}\tilde{u}_{\theta m} + m^2\frac{4G(\lambda+G)}{M}\frac{1}{r^2}\tilde{u}_{\theta m} - m\frac{2G}{M}\frac{1}{r}\tilde{\sigma}_m + m\frac{\lambda}{M}\frac{1}{r}\tilde{\sigma}_{zm},\end{aligned} \tag{5f}$$

$$\begin{aligned}\frac{\partial \tilde{\sigma}_m}{\partial z} = \frac{\gamma_w}{k}\tilde{Q}_m, \quad \frac{\partial \tilde{Q}_m}{\partial z} =& s\left[\frac{2G}{M}\left(\frac{\partial \tilde{u}_{rm}}{\partial r} + \frac{\tilde{u}_{rm}}{r}\right) + \frac{2G}{M} m \frac{1}{r} u_{\theta m} + \frac{1-2v}{2G(1-v)}(\tilde{\sigma}_{zm} + \tilde{\sigma}_m)\right] \\ & - \frac{k}{\gamma_w}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - m^2\frac{1}{r^2}\right)\tilde{\sigma}.\end{aligned} \tag{5g,h}$$

For the convenience of analysis, intermediate variables, $\tilde{u}_{vm}, \tilde{u}_{hm}, \tilde{\sigma}_{vzm}$ and $\tilde{\sigma}_{hzm}$, are introduced below:

$$\tilde{u}_{vm} = \tilde{u}_{rm} + \tilde{u}_{\theta m}, \quad \tilde{u}_{hm} = \tilde{u}_{rm} - \tilde{u}_{\theta m}, \quad \tilde{\sigma}_{vzm} = \tilde{\sigma}_{rzm} + \tilde{\sigma}_{\theta zm}, \quad \tilde{\sigma}_{hm} = \tilde{\sigma}_{rzm} - \tilde{\sigma}_{\theta zm}. \tag{6}$$

The $m$th-order Hankel transform and its inversion [34] are defined as:

$$\bar{f}(\xi, m, z, s) = \int_0^\infty \tilde{f}(r, \theta, z, s) J_m(\xi r) r \mathrm{d}r, \quad \tilde{f}(r, \theta, z, s) = \int_0^\infty \bar{f}(\xi, m, z, s) J_m(\xi r)\xi \mathrm{d}\xi,$$

where $\xi$ denotes the Hankel transform parameter, $J_m(\xi r)$ is the $m$th-order Bessel function.

Taking the Hankel transform to Eqs. (5a) + (5b), (5a) – (5b), (5c), (5d), (5e) + (5f), (5e) – (5f), (5g), and (5h), we obtain the following state-space equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{Bmatrix} \bar{u}_{vm} \\ \bar{u}_{hm} \\ \bar{u}_{zm} \\ \bar{\sigma}_{zm} \\ \bar{\sigma}_{vzm} \\ \bar{\sigma}_{hzm} \\ \bar{\sigma}_m \\ \bar{Q}_m \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \xi & 0 & \frac{1}{G} & 0 & 0 & 0 \\ 0 & 0 & -\xi & 0 & 0 & \frac{1}{G} & 0 & 0 \\ -\frac{\lambda}{2M}\xi & \frac{\lambda}{2M}\xi & 0 & \frac{1}{M} & 0 & 0 & \frac{1}{M} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\xi & \frac{1}{2}\xi & 0 & 0 \\ \frac{G(5\lambda+6G)}{2M}\xi^2 & -\frac{G(3\lambda+2G)}{2M}\xi^2 & 0 & \frac{\lambda}{M}\xi & 0 & 0 & -\frac{2G}{M}\xi & 0 \\ -\frac{G(3\lambda+2G)}{2M}\xi^2 & \frac{G(5\lambda+6G)}{2M}\xi^2 & 0 & -\frac{\lambda}{M}\xi & 0 & 0 & \frac{2G}{M}\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k'} \\ \frac{Gs}{M}\xi & -\frac{Gs}{M}\xi & 0 & \frac{s}{M} & 0 & 0 & k'\xi^2 + \frac{s}{M} & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_{vm} \\ \bar{u}_{hm} \\ \bar{u}_{zm} \\ \bar{\sigma}_{zm} \\ \bar{\sigma}_{vzm} \\ \bar{\sigma}_{hzm} \\ \bar{\sigma}_m \\ \bar{Q}_m \end{Bmatrix}, \tag{7}$$

where, $\bar{u}_{vm}, \bar{\sigma}_{vzm}$ are the $(m+1)$th-order Hankel transforms of $\tilde{u}_{vm}, \tilde{\sigma}_{vzm}$; $\bar{u}_{hm}, \bar{\sigma}_{hzm}$ are the $(m-1)$th-order Hankel transforms of $\tilde{u}_{hm}, \tilde{\sigma}_{hzm}$, and $\bar{u}_{zm}, \bar{\sigma}_{zm}, \bar{\sigma}_m, \bar{Q}_m$ are the $m$th-order Hankel transforms of $\tilde{u}_{zm}, \tilde{\sigma}_{zm}, \tilde{\sigma}_m, \tilde{Q}_m$; and $k' = \frac{k}{\gamma_w}$.

Equation 7 is an ordinary differential equation in form of a matrix which can be re-written as:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\bar{\boldsymbol{B}}(\xi, m, z, s)\right] = \boldsymbol{A}(\xi, s)\bar{\boldsymbol{B}}(\xi, m, z, s), \tag{8}$$

where $\bar{\boldsymbol{B}}(\xi, m, z, s) = [\bar{u}_{vm}, \bar{u}_{hm}, \bar{u}_{zm}, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{\sigma}_{hzm}, \bar{\sigma}_m, \bar{Q}_m]^{\mathrm{T}}$.

The solution to the ordinary-differential matrix equation (8) can be obtained using the theorem of the Cayley–Hamilton [32]; however, it will lead to a complex work load [22]. In this study, the solution of the ordinary-differential matrix equation (8) is obtained directly using the Laplace transform and its inversion [33] of the $z$-domain.

The Laplace transform and its inversion [33] of the $z$-domain are defined as:

$$\hat{f}(\xi, m, p, s) = \int_0^\infty \bar{f}(\xi, m, z, s)\mathrm{e}^{-pz}\mathrm{d}z, \quad \bar{f}(\xi, m, z, s) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(\xi, m, p, s)\mathrm{e}^{pz}\mathrm{d}p,$$

where $p$ denotes the Laplace-transform parameter of $z$.

Taking the Laplace transform of Eq. 8 with respect to the variable $z$ yields:

$$[\boldsymbol{I}p - \boldsymbol{A}(\xi, s)]\,\hat{\boldsymbol{B}}(\xi, m, p, s) = \bar{\boldsymbol{B}}(\xi, m, 0, s)\,, \tag{9}$$

where $\boldsymbol{I}$ is the $8 \times 8$ unit matrix, and $\hat{\boldsymbol{B}}(\xi, m, p, s)$ is the corresponding matrix of $\bar{\boldsymbol{B}}(\xi, m, z, s)$ in the Laplace-transform domain of $z$.

Applying the inversion of the Laplace transform with respect to $p$ yields:

$$\bar{\boldsymbol{B}}(\xi, m, z, s) = \boldsymbol{\Phi}(\xi, z, s)\bar{\boldsymbol{B}}(\xi, m, 0, s), \tag{10}$$

where $\boldsymbol{\Phi}(\xi, z, s)$ is the transfer matrix. This matrix establishes the relationship of $[\bar{u}_{vm}, \bar{u}_{hm}, \bar{u}_{zm}, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{\sigma}_{hzm}, \bar{\sigma}_m, \bar{Q}_m]^{\mathrm{T}}$ at the ground surface ($z = 0$) and an arbitrary depth $z$ in the transformed domain. The elements of the transfer matrix are listed in the Appendix.

Equation 10 can also be re-written as:

$$\bar{\boldsymbol{B}}(\xi, m, 0, s) = \boldsymbol{\Phi}(\xi, -z, s)\bar{\boldsymbol{B}}(\xi, m, z, s). \tag{11}$$

## 3 Solution of a multi-layered soil

Figure 1 shows that the soil medium has a total of $n$ layers of soil, and an arbitrary load is applied in the $m$th layer. Let $p_r(r, \theta, H_{m1}, t)$, $p_\theta(r, \theta, H_{m1}, t)$, and $q(r, \theta, H_{m1}, t)$ represent the load components in the $r$, $\theta$, and $z$ directions, respectively, and $H_{m1}$ is the depth from the ground surface to the loading plane. Let the thickness of the $i$th layer be $\Delta H_i = H_i - H_{i-1}$, in which $H_i$ and $H_{i-1}$ are the depths from the ground surface to the bottom and top of the $i$th layer, respectively.

Considering the loading surface as an artificial interface, we can use Eq. 10 for each layer which yields the following results:

$$\begin{cases} \bar{\boldsymbol{B}}(\xi, m, H_1^-, s) = \boldsymbol{\Phi}(\xi, \Delta H_1, s)\bar{\boldsymbol{B}}(\xi, m, 0, s) \\ \bar{\boldsymbol{B}}(\xi, m, H_2^-, s) = \boldsymbol{\Phi}(\xi, \Delta H_2, s)\bar{\boldsymbol{B}}(\xi, m, H_1^+, s) \\ \vdots \\ \bar{\boldsymbol{B}}(\xi, m, H_{m1}^-, s) = \boldsymbol{\Phi}(\xi, \Delta H_{m1}, s)\bar{\boldsymbol{B}}(\xi, m, H_{m-1}^+, s) \\ \bar{\boldsymbol{B}}(\xi, m, H_m^-, s) = \boldsymbol{\Phi}(\xi, \Delta H_{m2}, s)\bar{\boldsymbol{B}}(\xi, m, H_{m1}^+, s) \\ \vdots \\ \bar{\boldsymbol{B}}(\xi, m, H_n^-, s) = \boldsymbol{\Phi}(\xi, \Delta H_n, s)\bar{\boldsymbol{B}}(\xi, m, H_{n-1}^+, s), \end{cases} \tag{12}$$

where $\bar{\boldsymbol{B}}\left(\xi, m, H_i^+, s\right)$ is $[\bar{u}_{vm}, \bar{u}_{hm}, \bar{u}_{zm}, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{\sigma}_{hzm}, \bar{\sigma}_m, \bar{Q}_m]^{\mathrm{T}}$ of the $(i+1)$th layer when $z = H_i$, while $\bar{\boldsymbol{B}}\left(\xi, m, H_i^-, s\right)$ is $[\bar{u}_{vm}, \bar{u}_{hm}, \bar{u}_{zm}, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{\sigma}_{hzm}, \bar{\sigma}_m, \bar{Q}_m]^{\mathrm{T}}$ of the $i$th layer when $z = H_i$. $\Delta H_{m1} = H_{m1} - H_{m-1}$, and $\Delta H_{m2} = H_m - H_{m1}$.

In terms of the displacement compatibility and stress continuity at the interface between two adjacent layers, the following boundary conditions are obtained:

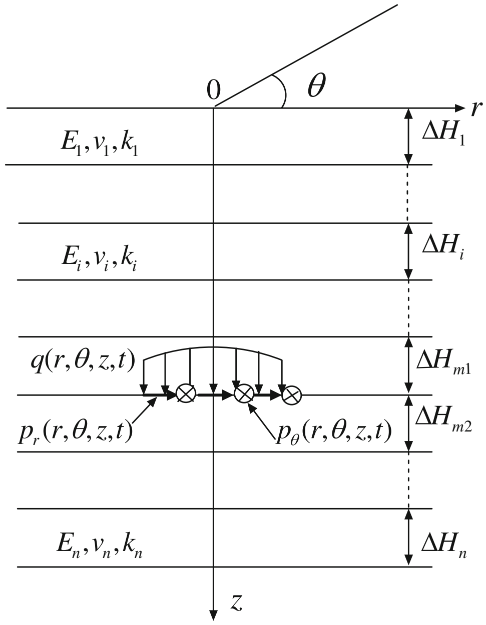


**Fig. 1** Geometry of a layered soil system

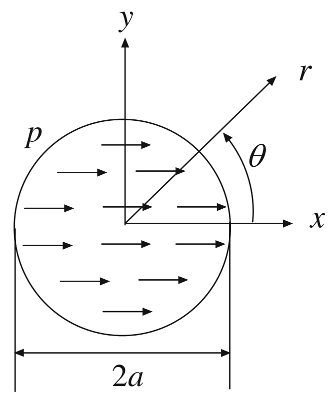


**Fig. 2** Circular tangential uniform loading on the surface

(1) at the natural interfaces:

$$\begin{cases} u_r(r,\theta,H_i^+,t) = u_r(r,\theta,H_i^-,t) \\ u_\theta(r,\theta,H_i^+,t) = u_\theta(r,\theta,H_i^-,t) \\ u_z(r,\theta,H_i^+,t) = u_z(r,\theta,H_i^-,t) \\ \sigma_z(r,\theta,H_i^+,t) = \sigma_z(r,\theta,H_i^-,t) \\ \sigma_{rz}(r,\theta,H_i^+,t) = \sigma_{rz}(r,\theta,H_i^-,t) \\ \sigma_{\theta z}(r,\theta,H_i^+,t) = \sigma_{\theta z}(r,\theta,H_i^-,t) \\ \sigma(r,\theta,H_i^+,t) = \sigma(r,\theta,H_i^-,t) \\ Q(r,\theta,H_i^+,t) = Q(r,\theta,H_i^-,t); \end{cases} \tag{13}$$

(2) at the artificial interfaces:

$$\begin{cases} u_r(r,\theta,H_{m1}^+,t) = u_r(r,\theta,H_{m1}^-,t) \\ u_\theta(r,\theta,H_{m1}^+,t) = u_\theta(r,\theta,H_{m1}^-,t) \\ u_z(r,\theta,H_{m1}^+,t) = u_z(r,\theta,H_{m1}^-,t) \\ \sigma_z(r,\theta,H_{m1}^+,t) = \sigma_z(r,\theta,H_{m1}^-,t) - q(r,\theta,H_{m1}^-,t) \\ \sigma_{rz}(r,\theta,H_{m1}^+,t) = \sigma_{rz}(r,\theta,H_{m1}^-,t) - p_r(r,\theta,H_{m1}^-,t) \\ \sigma_{\theta z}(r,\theta,H_{m1}^+,t) = \sigma_{\theta z}(r,\theta,H_{m1}^-,t) - p_\theta(r,\theta,H_{m1}^-,t) \\ \sigma(r,\theta,H_{m1}^+,t) = \sigma(r,\theta,H_{m1}^-,t) \\ Q(r,\theta,H_{m1}^+,t) = Q(r,\theta,H_{m1}^-,t). \end{cases} \tag{14}$$

Applying Eq. 4 to all the soil layers combined with the Laplace transform, the Fourier expansions and the Hankel transform of (13) and (14) yield the following equation:

$$\bar{\boldsymbol{B}}(\xi,m,H_n,s) = \boldsymbol{K}\cdot\bar{\boldsymbol{B}}(\xi,m,0,s) - \boldsymbol{\Psi}\cdot\boldsymbol{\Gamma}, \tag{15}$$

where,

$$\boldsymbol{K} = \boldsymbol{\Phi}(\xi, \Delta H_n, s)\boldsymbol{\Phi}(\xi, \Delta H_{n-1}, s)\ldots\boldsymbol{\Phi}(\xi, \Delta H_1, s),$$
$$\boldsymbol{\Psi} = \boldsymbol{\Phi}(\xi, \Delta H_n, s)\boldsymbol{\Phi}(\xi, \Delta H_{n-1}, s)\ldots\boldsymbol{\Phi}(\xi, \Delta H_{m2}, s),$$
$$\boldsymbol{\Gamma} = [0\ 0\ 0\ \bar{q}_m\ \ \bar{p}_{rm} + \bar{p}_{\theta m}\ \ \bar{p}_{rm} - \bar{p}_{\theta m}\ 0\ 0]^{\mathrm{T}}.$$

Equation 15 establishes the relationship between $\bar{\boldsymbol{B}}(\xi, m, 0, s)$ at the ground surface $(z = 0)$ and $\bar{\boldsymbol{B}}(\xi, m, H_n, s)$ at the bottom $(z = H_n)$. Equation 15 consists of $4 \times 4$ linear equations for the consolidation problem, which are independent of the number of soil layers. As a result, the unknown variables at the boundary can be obtained from the known ones at the boundary based on this relationship.

Supposing the ground surface is permeable, we have

$$\begin{aligned}&\sum_{m=0}^{\infty} \bar{\sigma}_{zm}(\xi, 0, s)\cos m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{\sigma}_{rzm}(\xi, 0, s)\cos m\theta = 0,\\ &\sum_{m=0}^{\infty} \bar{\sigma}_{\theta zm}(\xi, 0, s)\sin m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{\sigma}_{m}(\xi, 0, s)\cos m\theta = 0.\end{aligned} \tag{16}$$

Assuming the bottom of the soil system is fixed, we have that there are two possible drainage conditions:

(1) if the bottom is permeable, then

$$\begin{aligned}&\sum_{m=0}^{\infty} \bar{u}_{rm}(\xi, H_n, s)\cos m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{u}_{\theta m}(\xi, H_n, s)\sin m\theta = 0,\\ &\sum_{m=0}^{\infty} \bar{u}_{zm}(\xi, H_n, s)\cos m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{\sigma}_{m}(\xi, 0, s)\cos m\theta = 0;\end{aligned} \tag{17}$$

(2) if the bottom is impermeable, then

$$\begin{aligned}&\sum_{m=0}^{\infty} \bar{u}_{rm}(\xi, H_n, s)\cos m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{u}_{\theta m}(\xi, H_n, s)\sin m\theta = 0,\\ &\sum_{m=0}^{\infty} \bar{u}_{zm}(\xi, H_n, s)\cos m\theta = 0, \quad \sum_{m=0}^{\infty} \bar{Q}_{m}(\xi, H_n, s)\cos m\theta = 0.\end{aligned} \tag{18}$$

$\bar{\boldsymbol{B}}(\xi, m, 0, s)$ and $\bar{\boldsymbol{B}}(\xi, m, H_n, s)$ can be obtained from Eq. 15 by using the boundary conditions in (16–18). Hence, when a point at a given depth $z$ in the $i$th layer is above the loading surface, the variables in the transformed domain can be obtained by using Eq. 10:

$$\bar{\boldsymbol{B}}(\xi, m, z, s) = \boldsymbol{\Omega} \cdot \bar{\boldsymbol{B}}(\xi, m, 0, s), \tag{19}$$

in which, $\boldsymbol{\Omega} = \boldsymbol{\Phi}(\xi, z - H_{i-1}, s)\boldsymbol{\Phi}(\xi, \Delta H_{i-1}, s)\ldots\boldsymbol{\Phi}(\xi, \Delta H_1, s)$.

Similarly, when a point at a given depth $z$ in the $i$th layer is below the loading surface, the variables in the transformed domain can be obtained by using Eq. 11:

$$\bar{\boldsymbol{B}}(\xi, m, z, s) = \boldsymbol{\Theta} \cdot \bar{\boldsymbol{B}}(\xi, m, H_n, s), \tag{20}$$

where $\boldsymbol{\Theta} = \boldsymbol{\Phi}(\xi, z - H_i, s)\boldsymbol{\Phi}(\xi, -\Delta H_{i+1}, s)\ldots\boldsymbol{\Phi}(\xi, -\Delta H_n, s)$.

Equations 19 and 20 are the analytical solutions to the non-axisymmetric consolidation problems of a multi-layered soil in the transformed domain. Applying the inversion of the Hankel transform and the Laplace transform to (19) and (20) results in the solution in the physical domain.

For simplification, a circular vertical uniform load $q$ and a circular tangential uniform load $p$ (in the direction $x$) are assumed to be applied on the surface, respectively.

For a circular vertical uniform load $q$ applied on the ground surface, $q(r, \theta, 0, t) = q$, $(0 \le r \le a, 0 \le \theta \le 2\pi)$, the Laplace transform of the load $q$ yields:

$$\tilde{q}(r, \theta, 0, s) = \int_0^{\infty} q\mathrm{e}^{-st}\mathrm{d}t = \frac{q}{s}. \tag{21}$$

The Fourier expansions of the above equation yield:

$$\tilde{q}(r,\theta,0,s)=\sum_{m=0}^{\infty}\tilde{q}(r,m,0,s)\cos m\theta=\frac{q}{s}. \tag{22}$$

When $m=0$, $\tilde{q}(r,0,0,s)=q/s$.

The zero-order Hankel transform of equation $\tilde{q}(r,0,0,s)=q/s$ yields:

$$\bar{q}(\xi,0,0,s)=\int_0^{\infty}\frac{q}{s}J_0(\xi r)r\mathrm{d}r=\frac{qaJ_1(\xi a)}{\xi s}. \tag{23}$$

For a circular tangential uniform load $p$ (in the direction $x$, shown in Fig. 2) applied on the ground surface, we have

$$\begin{cases} p_r(r,\theta,0,t)=p\cos\theta\,, & (0\le r\le a, 0\le\theta\le 2\pi)\,,\\ p_\theta(r,\theta,0,t)=-p\sin\theta\,, & (0\le r\le a, 0\le\theta\le 2\pi), \end{cases} \tag{24}$$

where $a$ is the radius of the circle.

The Laplace transform of the above equation yields:

$$\begin{cases} \tilde{p}_r(r,\theta,0,s)=\int_0^{\infty}p\cos\theta\mathrm{e}^{-st}\mathrm{d}t=\dfrac{p\cos\theta}{s}\,, & (0\le r\le a, 0\le\theta\le 2\pi),\\ \tilde{p}_\theta(r,\theta,0,s)=-\int_0^{\infty}p\sin\theta\mathrm{e}^{-st}\mathrm{d}t=-\dfrac{p\sin\theta}{s}\,, & (0\le r\le a, 0\le\theta\le 2\pi). \end{cases} \tag{25}$$

The Fourier expansions of the above equation yield:

$$\tilde{p}_r(r,\theta,0,s)=\sum_{m=0}^{\infty}\tilde{p}_r(r,m,0,s)\cos m\theta=\frac{p\cos\theta}{s},$$

$$\tilde{p}_\theta(r,\theta,0,s)=\sum_{m=0}^{\infty}\tilde{p}_\theta(r,m,0,s)\sin m\theta=-\frac{p\sin\theta}{s}. \tag{26}$$

When $m=1$, $\tilde{p}_r(r,1,0,s)=-\tilde{p}_\theta(r,1,0,s)=p/s$.

In order to solve the problem, new intermediate variables $\tilde{p}_v$ and $\tilde{p}_h$ are introduced below

$$\tilde{p}_v=\tilde{p}_r(r,1,0,s)+\tilde{p}_\theta(r,1,0,s)=0,\quad \tilde{p}_h=\tilde{p}_r(r,1,0,s)-\tilde{p}_\theta(r,1,0,s)=\frac{2p}{s}. \tag{27}$$

The zero-order Hankel transform of equation $\tilde{p}_h=2p/s$ yields:

$$\bar{p}_h=\int_0^{\infty}\frac{2p}{s}J_0(\xi r)rdr=\frac{2paJ_1(\xi a)}{\xi s}. \tag{28}$$

## 4 Numerical results

In this study, the inversion of the Laplace transform is obtained by the numerical scheme proposed by Talbot [33]. The feasibility and efficiency of this scheme for consolidation problems were successfully demonstrated by Booker and Small [9–11]. The technique for implementing the inversion of the Hankel transform can be found in [21], and the feasibility and efficiency of this procedure were demonstrated by Ai et al. [21] and Chen [35–37]. Due to the existence of exponential-growth functions in the transfer-matrices, the conventional forward transfer-matrix method has an intrinsic fault, which would lead to ill-conditioned matrices for thick layers and accumulate numerical errors for a large number of layers. However, a backward transfer-matrix method suggested by Yue [38–40] can be used to overcome this ill-posedness associated with the conventional forward transfer-matrix method. Ai et al. [21] used a similar approach to overcome this problem. In this paper, the technique presented by Ai et al. [21] was used to eliminate the exponential-growth functions in the matrices and solutions.

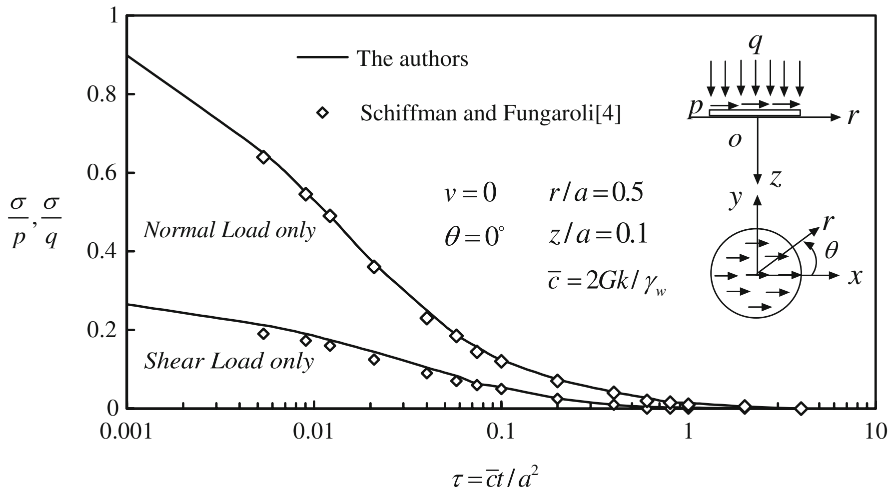


**Fig. 3** Consolidation of a semi-infinite soil medium

## 4.1 Semi-infinite soil

To verify the correctness and accuracy of the method proposed in this paper, the results for a semi-infinite soil medium provided by Schiffman and Fungaroli [4] are adopted and presented in Fig. 3. The comparisons show that the results obtained using the solutions from this study are in good agreement with those by Schiffman and Fungaroli [4].

## 4.2 Two-layered soil

The results for a two-layered soil obtained by the finite-layer method [10] are presented in Fig. 4 as compared with those obtained using the solution from this study. These results were based on a uniform vertical surface load applied on the ground surface in a circular area. The comparison clearly shows that the results from this study match well those by Booker and Small [10]. Figure 4 also shows that the vertical displacement at the center of the loading area on the ground surface increases with the time factor, $\tau$.

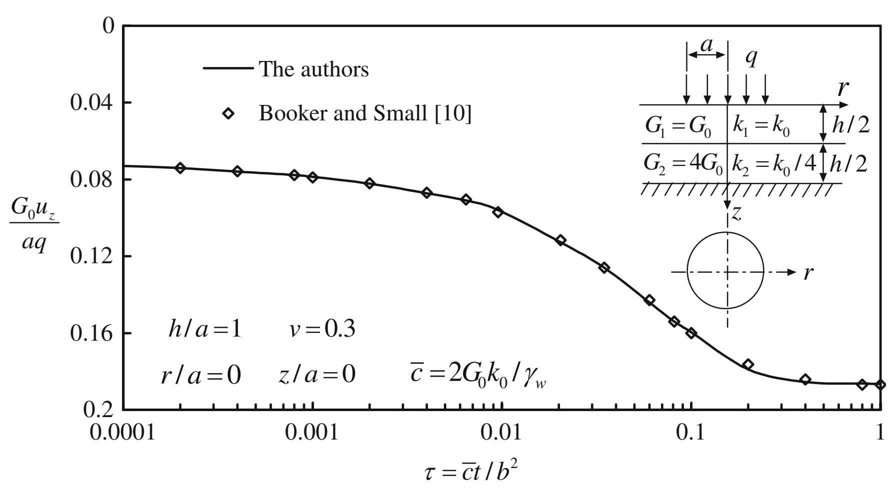


**Fig. 4** Consolidation of a two-layered soil

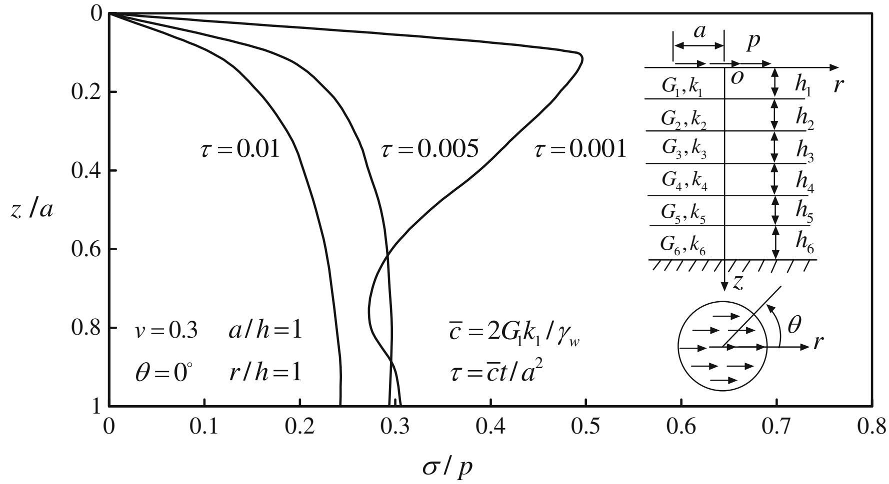


**Fig. 5** Consolidation of a six-layered soil

### 4.3 Multi-layered soil

In practice, soils on sites are mostly multi-layered, in which each soil layer may have different soil type and physical and mechanical properties. A six-layered soil profile as shown in Fig. 5 is selected in this study to demonstrate the use of the proposed method for the consolidation of a multi-layered soil. It is assumed that the surface is permeable and the base is fixed and impermeable. The shear modulus, the thickness, and the permeability of each soil layer have the following relationships, i.e.,

$$G_1{:}G_2{:}G_3{:}G_4{:}G_5{:}G_6 = 1{:}4{:}2{:}4{:}2{:}1,$$
$$k_1{:}k_2{:}k_3{:}k_4{:}k_5{:}k_6 = 1{:}2{:}4{:}2{:}1{:}2,$$
$$h_1{:}h_2{:}h_3{:}h_4{:}h_5{:}h_6 = 1{:}2{:}1{:}4{:}1{:}1.$$

The total soil thickness is $h = h_1 + h_2 + h_3 + h_4 + h_5 + h_6$. In this computation, each soil layer is assumed to have identical Poisson ratio of 0.3. The excess pore-water pressure distributions at different time factors are presented in Fig. 5. It is shown that the excess pore-water pressure at the beginning is not uniformly distributed due to different moduli and a peak value exists at a depth of approximately $0.1a$. With the increase of time, the excess pore-water pressure is more uniformly distributed and decreases in magnitude.

## 5 Conclusions

In this study, analytical solutions to the problem concering the consolidation of a multi-layered soil under circular loading have been obtained using the transfer-matrix method. Starting from the governing equations of a saturated poroelastic soil in a cylindrical coordinate system, a set of eighth-order state-space equations of Biot's consolidation has been obtained by eliminating the variation of time $t$ through the Laplace transform, Fourier expansions with respect to the coordinate $\theta$, and the Hankel transform with respect to the coordinate $r$. The solution of the eighth-order state-space equation was derived directly using the Laplace transform and its inversion of the $z$-domain. Based on the continuity conditions between layers and the boundary conditions, the analytical solutions for the consolidation of a multi-layered soil under circular loading in the transformed domain were obtained by utilizing the transfer-matrix method. The final actual solutions in the physical domain could be obtained by the inversion of the Laplace transform and the Hankel transform. Numerical calculations were carried out by a corresponding program for a semi-infinite soil medium, two-layered soil, and six-layered soil. Comparisons for the former two cases show that the solutions proposed in this study yield almost the same results as those obtained by Schiffman and Fungaroli [4] and Booker and Small [10]. The numerical analysis of the six-layered soil shows the high accuracy and efficiency of the present solutions in solving for the consolidation of a multi-layered soil under circular loading.

The analytical solutions presented in this paper can be used to evaluate the consolidation behavior of a multi-layered soil under circular loading, such as circular building foundations, oil tanks, and silos etc. In addition, the analytical solutions proposed in this paper can be used as fundamental solutions for the interaction analysis between a saturated multi-layered soil and foundation structures by combining with the boundary-element method. Poro-elastic media have been assumed in this study. Under the assumption of an isotropic and linearly elastic material, the constitutive law of soil in tension, compression and shearing should be the same. In reality, however, soils are often anisotropic and have nonlinear and plastic behavior; therefore, their constitutive laws in tension, compression, and shearing may be different and very complex. This complex nature of soils will be considered in a future study.

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## Appendix

$$\Phi_{11} = \frac{G\xi^2 C}{Ms}(\cosh\xi z - \cosh qz) + \frac{1}{2}\xi z\sinh\xi z + \cosh\xi z = \Phi_{22} = \Phi_{55} = \Phi_{66},$$

$$\Phi_{12} = -\frac{G\xi^2 C}{Ms}(\cosh\xi z - \cosh qz) - \frac{1}{2}\xi z\sinh\xi z = \Phi_{21} = \Phi_{56} = \Phi_{65},$$

$$\Phi_{13} = \xi z\cosh\xi z + \frac{2G\xi^2 C}{Msq}(q\sinh\xi z - \xi\sinh qz) = -\Phi_{23}, \quad \Phi_{14} = \frac{\xi C}{Ms}(\cosh\xi z - \cosh qz) + \frac{z}{2G}\sinh\xi z$$
$$= -\Phi_{24},$$

$$\Phi_{15} = \frac{1}{4G\xi}(\xi z\cosh\xi z + 3\sinh\xi z) + \frac{\xi C}{2Msq}(q\sinh\xi z - \xi\sinh qz) = \Phi_{26},$$

$$\Phi_{16} = -\frac{1}{4G\xi}(\xi z\cosh\xi z - \sinh\xi z) - \frac{\xi C}{2Msq}(q\sinh\xi z - \xi\sinh qz) = \Phi_{25},$$

$$\Phi_{17} = \frac{\xi C}{Ms}(\cosh\xi z - \cosh qz) = -\Phi_{27}, \quad \Phi_{18} = \frac{1}{qs}(q\sinh\xi z - \xi\sinh qz) = -\Phi_{28},$$

$$\Phi_{31} = \frac{G\xi^2 C}{qMs}(\xi\sinh qz - q\sinh\xi z) + \frac{G\xi}{qM}\sinh qz - \frac{1}{2}\xi z\cosh\xi z = -\Phi_{32},$$

$$\Phi_{33} = -\frac{2G\xi^2 C}{Ms}(\cosh\xi z - \cosh qz) - \xi z\sinh\xi z + \cosh\xi z = \Phi_{44},$$

$$\Phi_{34} = -\frac{\xi C}{Msq}(q\sinh\xi z - \xi\sinh qz) - \frac{1}{2G\xi}(\xi z\cosh\xi z - \sinh\xi z) + \frac{1}{Mq}\sinh qz,$$

$$\Phi_{35} = -\frac{\xi C}{2Ms}(\cosh\xi z - \cosh qz) - \frac{z}{4G}\sinh\xi z = -\Phi_{36},$$

$$\Phi_{37} = -\frac{\xi C}{Msq}(q\sinh\xi z - \xi\sinh qz) + \frac{1}{Mq}\sinh qz, \quad \Phi_{38} = \frac{1}{s}(\cosh qz - \cosh\xi z),$$

$$\Phi_{41} = -\frac{2G^2\xi^3 C}{Ms}(\cosh\xi z - \cosh qz) - G\xi^2 z\sinh\xi z = -\Phi_{42}, \quad \Phi_{43} = \frac{4G^2\xi^3 C}{Msq}(\xi\sinh qz - q\sinh\xi z)$$
$$+ 2G\xi(\sinh\xi z - \xi z\cosh\xi z),$$

$$\Phi_{45} = \frac{G\xi^2 C}{qMs}(\xi\sinh qz - q\sinh\xi z) - \frac{1}{2}\xi z\cosh\xi z = -\Phi_{46},\quad \Phi_{47} = -\frac{2G\xi^2 C}{Ms}(\cosh\xi z - \cosh qz),$$

$$\Phi_{48} = \frac{2G\xi}{qs}(\xi\sinh qz - q\sinh\xi z),\quad \Phi_{51} = -\frac{2G^2\xi^3 C}{Msq}(\xi\sinh qz - q\sinh\xi z) + \frac{1}{2}G\xi(3\sinh\xi z + 2\xi z\cosh\xi z) - \frac{2G^2\xi^2}{Mq}\sinh qz = \Phi_{62},$$

$$\Phi_{52} = \frac{2G^2\xi^3 C}{Msq}(\xi\sinh qz - q\sinh\xi z) - \frac{1}{2}G\xi(\sinh\xi z + 2\xi z\cosh\xi z) + \frac{2G^2\xi^2}{Mq}\sinh qz = \Phi_{61},$$

$$\Phi_{53} = \frac{4G^2\xi^3 C}{Ms}(\cosh\xi z - \cosh qz) + 2G\xi^2 z\sinh\xi z = -\Phi_{63},$$

$$\Phi_{54} = \frac{2G\xi^2 C}{Msq}(q\sinh\xi z - \xi\sinh qz) + \xi z\cosh\xi z - \frac{2G\xi}{Mq}\sinh qz = -\Phi_{64},$$

$$\Phi_{57} = -\frac{2G\xi^2 C}{Msq}(\xi\sinh qz - q\sinh\xi z) - \frac{2G\xi}{Mq}\sinh qz = -\Phi_{67},$$

$$\Phi_{58} = \frac{2G\xi}{s}(\cosh\xi z - \cosh qz) = -\Phi_{68},\quad \Phi_{71} = -G\xi(\cosh\xi z - \cosh qz) = -\Phi_{72},$$

$$\Phi_{73} = -\frac{2G\xi}{q}(q\sinh\xi z - \xi\sinh qz),$$

$$\Phi_{74} = \cosh qz - \cosh\xi z,\quad \Phi_{75} = -\frac{1}{2q}(q\sinh\xi z - \xi\sinh qz) = -\Phi_{76},\quad \Phi_{77} = \cosh qz = \Phi_{88},$$

$$\Phi_{78} = \frac{M}{Cq}\sinh qz,\quad \Phi_{81} = \frac{G\xi^2 C}{qM}(\xi\sinh qz - q\sinh\xi z) + \frac{Gs\xi}{qM}\sinh qz = -\Phi_{82},$$

$$\Phi_{83} = -\frac{2G\xi^2 C}{M}(\cosh\xi z - \cosh qz),\quad \Phi_{84} = \frac{\xi C}{qM}(\xi\sinh qz - q\sinh\xi z) + \frac{s}{qM}\sinh qz,$$

$$\Phi_{85} = -\frac{\xi C}{2M}(\cosh\xi z - \cosh qz) = -\Phi_{86},\quad \Phi_{87} = \frac{qC}{M}\sinh qz,$$

where $C = \frac{k}{\gamma_w}M$, and $q^2 = \xi^2 + \frac{s}{C}$.

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